Cramer's Rule and Determinants

$$\det\begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{matrix} 1 & 2 \\ -2 & 0 \\ 1 & 3 \end{matrix}$$

$$[(1\cdot 0\cdot 2)+(2\cdot 3\cdot 1)+(1\cdot -2\cdot 3)]-[(1\cdot 0\cdot 1)+(3\cdot 3\cdot 1)+(2\cdot -2\cdot 2)]$$

$$(0+6-6)-(0+9-8)=-1$$

Cramer's rule is another method for solving linear systems.

Cramer's rule for a 2 × 2 System: Let A be the coefficient matrix of the linear system $ax + by = e$ and $cx + dy = f$. If $\det A \neq 0$, then the system has exactly one solution.

The solution is $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det A}$ and $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A}$.

Cramer's rule for a 3 × 3 System: Let A be the coefficient matrix of the linear system $ax + by + cz = j$, $dx + ey + fz = k$ and $gx + hy + iz = \ell$. If $\det A \neq 0$, then the system has exactly one solution.

The solution is $x = \dfrac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\det A}$, $y = \dfrac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\det A}$, and $z = \dfrac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\det A}$.

$$2x + y = 11$$
$$-3x + 2y = 1$$

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\det \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} = 2(2) - (-3)(1) = 7$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det A} = \frac{\begin{vmatrix} 11 & 1 \\ 1 & 2 \end{vmatrix}}{7} = \frac{11(2) - 1(1)}{7} = \frac{21}{7} = 3$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A} = \frac{\begin{vmatrix} 2 & 11 \\ -3 & 1 \end{vmatrix}}{7} = \frac{2(1) - (-3)(11)}{7} = \frac{35}{7} = 5$$

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\det A}, \; y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\det A}, \text{ and } z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\det A}.$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{aligned} -6x + 4y + z &= 32 \\ 5x + 2y + 3z &= 13 \\ x - y + z &= -5 \end{aligned} \qquad A = \begin{bmatrix} -6 & 4 & 1 \\ 5 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \qquad \det \begin{bmatrix} -6 & 4 & 1 \\ 5 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix} = -45$$

$$x = \frac{\begin{vmatrix} 32 & 4 & 1 \\ 13 & 2 & 3 \\ -5 & -1 & 1 \end{vmatrix}}{-45} = \frac{45}{-45} = -1 \qquad y = \frac{\begin{vmatrix} -6 & 32 & 1 \\ 5 & 13 & 3 \\ 1 & -5 & 1 \end{vmatrix}}{-45} = \frac{-270}{-45} = 6$$

$$z = \frac{\begin{vmatrix} -6 & 4 & 32 \\ 5 & 2 & 13 \\ 1 & -1 & -5 \end{vmatrix}}{-45} = \frac{-90}{-45} = 2$$