Mathematical Induction

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Let's start with an example:

$S_1 = 1 = 1^2$
$S_2 = 1 + 3 = 2^2$
$S_3 = 1 + 3 + 5 = 3^2$
$S_4 = 1 + 3 + 5 + 7 = 4^2$

If we are to make a conclusion from this pattern then we can conclude that the sum of the first n odd integers is
$S_n = 1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$
We need a way to prove this works for all odd integers.

This method is called Induction.

Induction
Let P_n be a statement involving the postive integer, n. If
1. P_1 *is true*
2. The truth of P_k implies the truth of P_{k+1}, for every postitive integer k, then P_nMust be true for all positive integers n.

This is best learned by working through examples.

Preliminary Example: Finding the k+1 term.

a). $P_k: S_k = \frac{k^2(k+1)^2}{4}$

$$P_{k+1}: S_{k+1} = \frac{(k+1)^2(k+1+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

b). $P_k: S_k = 1 + 5 + 9 + \ldots + [4(k-1) - 3] + (4k - 3)$

$$P_{k+1}: S_{k+1} = 1 + 5 + 9 + \ldots + [4(k-1+1) - 3] + [4(k+1) - 3]$$
$$P_{k+1}: S_{k+1} = 1 + 5 + 9 + \ldots + [4k - 3] + (4k + 1)$$

Let's try using Induction now.
Remember: $S_n = 1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$

Step 1. Show that the formula actually works for n=1
$S_n = 1 = 1^2 \quad True$

Step 2. Plug in k for n. (this step is mostly a formality).

$S_k = 1 + 3 + 5 + 7 + \ldots + (2k - 1) = k^2$

Step 3. Show it is true for the k+1 term. Here is where most of the work happens.

$$S_{k+1} = 1 + 3 + 5 + 7 + \ldots + (2k-1) + (2(k+1)-1) = (k+1)^2 \quad True\ or\ not???$$

Simplify the left hand side: **MAKE A KEY SUBSTITUTION!**

$$S_{k+1} = 1 + 3 + 5 + 7 + \ldots + (2k-1) + (2k+1)$$

$S_{k+1} = k^2 + (2k+1)$ This is why we did step 2.

Now simplify. In this example we are going to factor.

$$S_{k+1} = k^2 + 2k + 1 = (k+1)^2$$

Because we were able to make the left side equal the right we have proven the statement true by induction.

One last example.

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Step 1. Plug in 1 to verify it actually works for the first integer.

$$S_1 = 1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1 \quad True$$

Step 2. Plug k into the problem.

Step

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

$$S_{k+1} = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)\big((k+1)+1\big)(2(k+1)+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Make that key replacement from step 2.

$$S_{k+1} = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$S_{k+1} = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Simplify the left side.

$S_{k+1} = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$ Factor out the k+1 term on top.

$$S_{k+1} = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

Continue to simplify.

$$S_{k+1} = \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} = \frac{(k+1)[2k^2 + 7k + 6]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

Therefore we have proven it true by induction.